

# The survival of quantum coherence in deformed states superposition

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We study the dissipative dynamics of deformed coherent states superposition. We find that such kind of superposition can be more robust against decoherence than the usual Schrödinger cat states.

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The feature of quantum mechanics which most distinguishes it from classical mechanics is the coherent superposition of distinct physical states. Many of the less intuitive aspects of the quantum theory can be traced to this feature. The famous Schrödinger cat argument [1] highlights problems of interpretation where macroscopic superposition states is allowed. In fact, such states are very fragile in the presence of dissipation, and rapidly collapse to a classical mixture exhibiting no unusual interference features [2,3]. Environment induced decoherence has been identified in recent years as one of the main ingredients in the transition from quantum to classical behavior [4,5]. Classicality emerges as a consequence of the coupling of quantum systems to an environment which, in effect, dynamically enforces super-selection rules by precluding the stable existence of the majority of states in the Hilbert space of the system. The physics of decoherence has been studied during the last few years both from the theoretical [5] and also from the experimental point of view [6]. However, the most used paradigmatic case has been the superposition of two distinguishable coherent states [7]. The latter being eigenstates of boson annihilation operator [8].

On the other hand, quantum groups [9], introduced as a mathematical description of deformed Lie algebras, have given the possibility of generalizing the notion of creation and annihilation operators of the usual oscillator and to introduce deformed oscillator [10]. They were interpreted [11] as nonlinear oscillators with a very specific type of nonlinearity, and this led to the more general concept of  $f$ -deformed oscillator [12]. Then, the notion of  $f$ -coherent states was straightforwardly introduced [12], and the generation of such nonlinear coherent states enters in the real possibilities of trapped systems [13].

Successively, it was quite natural to consider the superposition of such states [14], which could be named *deformed cat state*. Here, we are going to study their dissipative dynamics and the related coherence properties in comparison with the usual Schrödinger cat states [7]. In particular, we shall show that the deformed cat states may result much more robust against decoherence than their undeformed version.

The essential point in understanding quantum coherence is the physical distinction between the coherent superposition state

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle \iff \rho = \sum_{i,j} c_i c_j^* |\psi_i\rangle \langle \psi_j|, \quad (1)$$

and the classical mixture

$$\rho_{mix} = \sum_i |c_i|^2 |\psi_i\rangle \langle \psi_i|. \quad (2)$$

Following the avenue sketched in Ref. [3], the density operator (1) can be written as

$$\rho = \rho_{mix} + \sum_{i \neq j} c_i c_j^* |\psi_i\rangle \langle \psi_j|. \quad (3)$$

Let  $Z$  be the operator corresponding to some physical observable having eigenvalues  $z$ . Then, the probability distribution for  $Z$  in the state  $|\psi\rangle$  is given by

$$P(z) = P_{mix}(z) + \sum_{i \neq j} c_i c_j^* \langle z | \psi_i \rangle \langle \psi_j | z \rangle, \quad (4)$$

where  $P_{mix}(z) = \sum_i |c_i|^2 |\langle z | \psi_i \rangle|^2$ . Measurement of  $Z$  will distinguish the states  $\rho$  and  $\rho_{mix}$  provided the second term in (4) is nonzero. We are thus led to define the quantum coherence function (with respect to the measurement  $Z$ ) as

$$\mathcal{C}(z) = \sum_{i \neq j} c_i c_j^* \langle z | \psi_i \rangle \langle \psi_j | z \rangle. \quad (5)$$

The usual coherent state is defined [8] as the eigenstate of the annihilation operator  $a$ , of a bosonic field

$$|\alpha\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad N = [\exp(\alpha^2)]^{-1/2}, \quad (6)$$

where we have assumed  $\alpha \in \mathbf{R}$  for the sake of simplicity. Then, it is possible to consider the superposition

$$|\Psi\rangle = N_+ (|\alpha\rangle + |-\alpha\rangle), \quad N_+ = [2 + 2N^2 \exp(-\alpha^2)]^{-1/2}, \quad (7)$$

known as *even cat state* Ref. [7]. This is a remarkable example of interference among quantum states.

Now, consider to measure the number operator, i.e.  $Z \equiv a^\dagger a$ , then, the number probability distribution results

$$P(n) = P_+(n) + P_-(n) + 2\text{Re}\{\mathcal{C}(n)\}, \quad (8)$$

with  $P_\pm(n) = |\langle n| \pm \alpha\rangle|^2$  and  $\mathcal{C}(n) = \langle \alpha|n\rangle \langle n| - \alpha\rangle$ . Hence, a convenient measure of quantum coherence is the quantum visibility

$$\mathcal{V} = \frac{|\mathcal{C}(n)|}{\sqrt{P_+(n)P_-(n)}}. \quad (9)$$

A straightforward calculation gives  $\mathcal{V} = 1$ , which means that the state (7) shows maximum quantum coherence.

In order to apply the same arguments at macroscopic level, it is also useful to define a parameter giving the separation among the two states being superposed. However, the concept of “distance” between quantum states is not uniquely defined (a recent discussion on this problem can be found in Ref. [15]), and here, we adopt the simplest one, leading to

$$d = \langle \alpha| (a + a^\dagger) |\alpha\rangle = 2\alpha. \quad (10)$$

It is worth noting that such distance is directly related to the number of photons  $\alpha^2$  of the coherent states.

We now introduce the decoherence effects due to a dissipative interaction with an environment [5]. This can be described (in interaction picture) by the following master equation of the Lindblad form [16]

$$\dot{\rho} = \gamma a \rho a^\dagger - \frac{\gamma}{2} \{a^\dagger a, \rho\}, \quad (11)$$

where  $\gamma$  is the damping rate, and we have set the bath temperature equal to zero. The decoherence effect on the state  $\rho(0) = |\Psi\rangle\langle\Psi|$  can be described in the following way [17]

$$\rho(t) = \sum_{k=0}^{\infty} \Upsilon_k(t) \rho(0) \Upsilon_k^\dagger(t), \quad (12)$$

where

$$\Upsilon_k(t) = \sum_{n=k}^{\infty} \sqrt{\binom{n}{k}} [\eta(t)]^{(n-k)/2} [1 - \eta(t)]^{k/2} |n-k\rangle\langle n|, \quad (13)$$

with  $\eta(t) = e^{-\gamma t}$ .

Thus, the quantum visibility results

$$\mathcal{V}(n, t) = \exp\{-2\alpha^2 [1 - \eta(t)]\}. \quad (14)$$

This is a well known results [3,18] showing that the decoherence effect depends on the damping rate as well as on the separation of the coherent states, i.e. the macroscopicity. Moreover,  $\mathcal{V}$  remains constant through  $n$ . Considering that  $\gamma$  is fixed by the system-environment interaction, we would investigate whether the quantum visibility can depend on the type of cat state one consider.

Let us introduce a  $f$ -coherent state defined [12] as the eigenstate of the annihilation operator of a  $f$ -deformed bosonic field  $A = a\sqrt{f(a^\dagger a)}$ , where  $f$  is an operator-valued function of the number operator (here it is assumed Hermitian and real). In general, it can be made dependent on continuous parameters, in such a way that, for given particular values, the usual algebra is recovered. The  $f$ -coherent state can be written as

$$|\zeta, f\rangle = \mathcal{N} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{[n]_f!}} |n\rangle, \quad \mathcal{N} = [\exp_f(\zeta^2)]^{-1/2}, \quad (15)$$

where we have considered  $\zeta \in \mathbf{R}$ , and we have introduced

$$\exp_f(x) \equiv \sum_{n=0}^{\infty} \frac{x^n}{[n]_f!}, \quad [n]_f! \equiv [nf(n)] \times [(n-1)f(n-1)] \times \dots [2f(2)] \times [f(1)] \times [f(0)]. \quad (16)$$

The function  $\exp_f$  is a deformed version of the usual exponential function. They become coincident when  $f$  is the identity. Notice that  $\exp_f(x) \exp_f(y) \neq \exp_f(x+y)$ , i.e. we have a non-extensive exponential which can be found in many physical problems [19].

Let us now consider the superposition

$$|\Phi\rangle = \mathcal{N}_+ (|\zeta, f\rangle + |-\zeta, f\rangle), \quad \mathcal{N}_+ = [2 + 2\mathcal{N}^2 \exp_f(-\zeta^2)]^{-1/2}. \quad (17)$$

In this case the separation between the two superposed states becomes

$$d = \langle \zeta, f | (a + a^\dagger) |\zeta, f\rangle = \mathcal{N}^2 \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{[n]_f!}} \left\{ \frac{\sqrt{n} \zeta^{(n-1)}}{\sqrt{[n-1]_f!}} + \frac{\sqrt{n+1} \zeta^{(n+1)}}{\sqrt{[n+1]_f!}} \right\}. \quad (18)$$

The decoherence effects on the state  $\rho(0) = |\Phi\rangle\langle\Phi|$  are introduced by again employing Eq.(12). Here, we are not interested on the dynamics in presence of a deformed Hamiltonian [20]. Then, the quantum visibility can be obtained as straightforward extension of previous argument

$$\mathcal{V}(n, t) = \left| \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{[-\zeta^2 (1 - \eta(t))]^k}{[n+k]_f!} \right| \times \left\{ \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{[\zeta^2 (1 - \eta(t))]^k}{[n+k]_f!} \right\}^{-1}. \quad (19)$$

From the above equation, it is clear that the quantum visibility depends on the outcome of the measurement of the observable  $Z$ . Moreover  $\mathcal{V}$ , depending on the specific form of  $f$ , could be greater than the non-deformed case. It follows the possibility to preserve the quantum coherence. Of course, in order to correctly compare the two situations, i.e. deformed and undeformed, one should consider cat states of the same macroscopicity, i.e. having the same  $d$ .

Then, let us consider two types of deformations in more details. The  $q$ -deformation defined by [10,11]

$$f(n) = \sqrt{\frac{1}{n} \frac{q^n - q^{-n}}{q - q^{-1}}}, \quad f(0) = 1, \quad q \in \mathbf{R}, \quad (20)$$

and the deformation given by [13],

$$f(n) = \frac{L_n^1(\xi^2)}{(n+1)L_n^0(\xi^2)}, \quad \xi \in \mathbf{R}, \quad (21)$$

which we are going to name  $L$ -deformation, since  $L_n^m$  indicates the associate Laguerre polynomial. It is worth noting that such  $L$ -deformation naturally arises in ion trapped systems [13].

From Eq.(14) we can argue a survival of coherence by decreasing  $\zeta$  with respect to  $\alpha$  still maintaining the same states separation  $d$ . In Fig.(1) we show the separation as function of deformation parameter when  $\zeta^2 = \alpha^2 = 2$ . The dashed line represents the value of undeformed cat state. Then, we may see that the  $q$ -deformation (dotted line) always implies a smaller distance  $d$ , while the  $L$ -deformation (solid line) allows to reach, and overcome, the distance of the undeformed coherent states ( $2\sqrt{2}$ ). This is due to the form of Eq.(21) which shows many singularities by varying  $\xi$ . Thus, for our purpose, the  $L$ -deformation seems more pertinent since allows a comparison between deformed and undeformed cat states having the same separation.

In Fig.(2) we show the quantum visibility as function of dimensionless time  $\gamma t$ . We immediately see that the decoherence of  $L$ -deformed cat state (solid lines) is slowed down with respect to the undeformed one (dashed line). Moreover, in the deformed case the quantum visibility depends on  $n$ , and it is better preserved by increasing  $n$ . Of course the relevant value is  $n = 2$  (since it is the most probable for both case) and it shows a consistent improvement.

Nevertheless, as stated above, we expect a better result for  $\zeta < \alpha$ . Then, in Fig.(3), we have plotted the quantum visibility at  $\gamma t = 1$  as function of  $\zeta$ . For each value of  $\zeta$  we have used the value of  $\xi$  giving the same distance  $d$  of the

undeformed case ( $2\sqrt{2}$ ). We may see that there exist a value of  $\zeta^2$  which maximize the coherence persistence. For such a value ( $\zeta^2 \approx 1$ ), the visibility results enormous greater than the undeformed case. For  $\zeta^2 \rightarrow 0$  the visibility tends to zero since practically the two superposing states become the vacuum states. Instead, for  $\zeta^2 \gg 2$ , the deformed case has always a distance  $d$  greater than the deformed one, so that no comparison is possible.

In conclusion we have shown that quantum coherence can survive against dissipation provided to superpose distinguishable coherent states of suitable deformed field. This unexpected result relies on the fact that the states being superposed, once deformed, are no longer eigenstates (nor near eigenstates) of the operator appearing in the irreversible part of the evolution equation (11), i.e. they substantially differ from the pointer basis [5]. On the other hand, deformed states, due to their nonlinear character, give rise to a more rich phase space structure [13], part of which can easier survive against decoherence.

The present results may open new perspectives for the experimental observation of macroscopic realism in quantum mechanics. Moreover, the kind of studied states, being decoherence resistant, could result quite useful for quantum information processing [21]. Extension of the above arguments to other observables, or to other types of decoherence, e.g. non-dissipative decoherence [22], is planned for future work.

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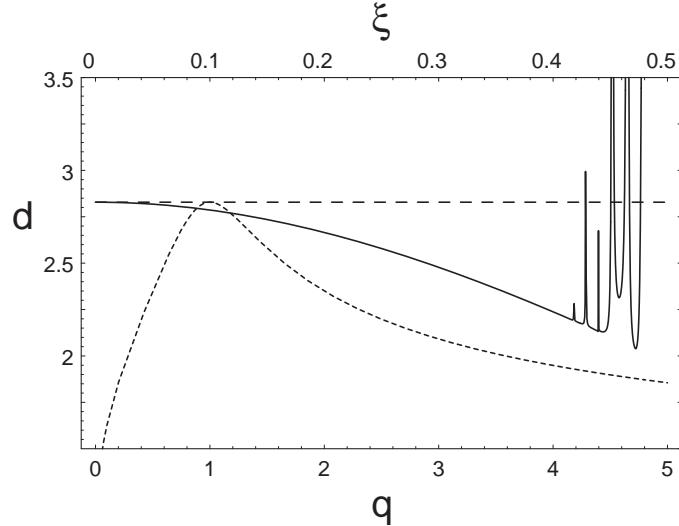


FIG. 1. States separation versus parameter deformation. The dashed line represents the non-deformed case; the dotted line the  $q$ -deformation; the solid line the  $L$ -deformation.  $\zeta^2 = \alpha^2 = 2$ .

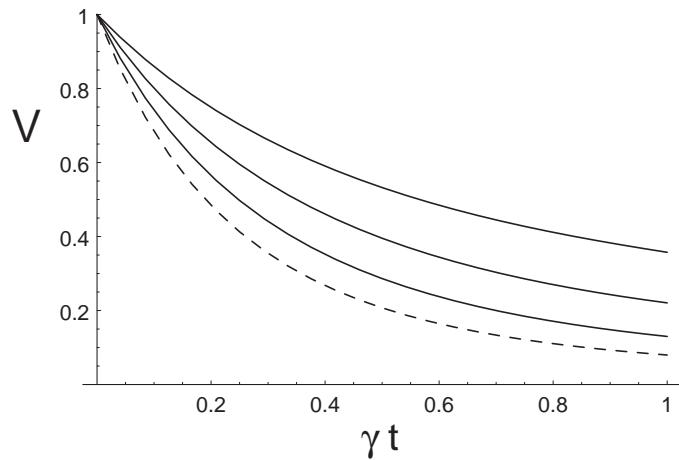


FIG. 2. Quantum visibility as function of dimensionless time  $\gamma t$ . The dashed line represents the non-deformed case; the solid lines the  $L$ -deformed case with  $\zeta^2 = \alpha^2 = 2$  and  $\xi = 0.45048$ . From top to bottom solid lines refer to  $n = 3$ ,  $n = 2$  and  $n = 1$ .

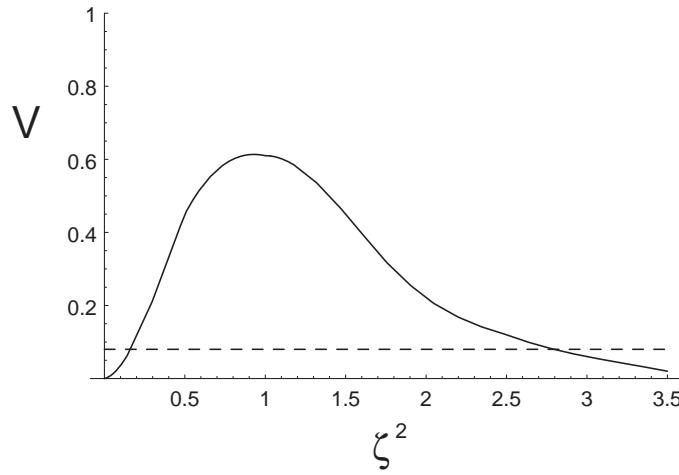


FIG. 3. Quantum visibility at  $\gamma t = 1$  as function of  $\zeta^2$ . The dashed line represents the non-deformed case with  $\alpha^2 = 2$ ; the solid lines refers to  $n = 2$ , for  $L$ -deformed case.